ME 323: FLUID MECHANICS-II

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Lecture-04

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Perfect-Gas Area Changes

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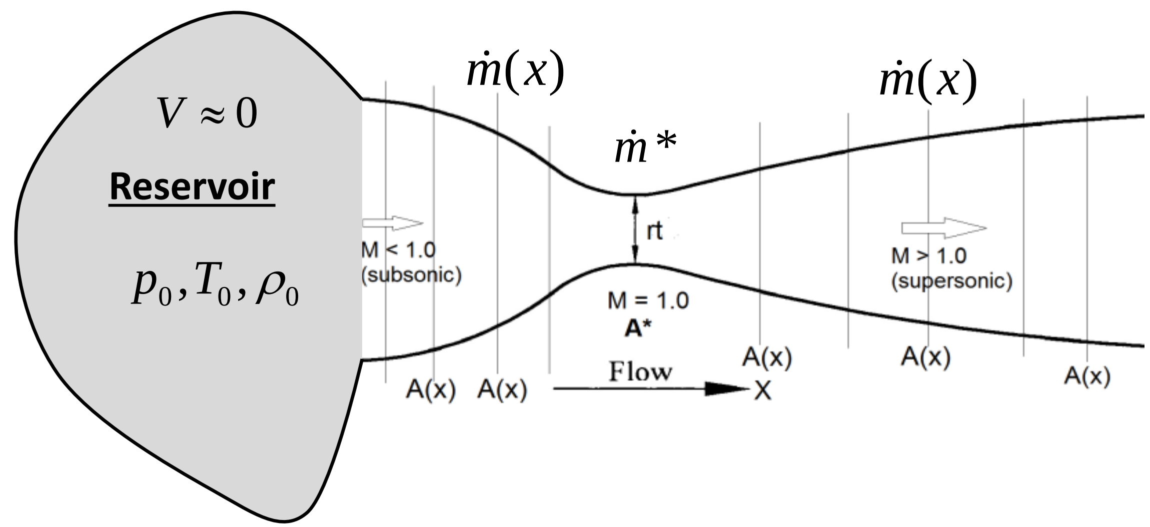
Relation for local area, *A*(x) and local Mach number, *M*(x)

The perfect-gas and isentropic relations can be used to convert the continuity equation into an algebraic expression involving only area and Mach number. Consider the mass flow at any section, *A*(*x*) to the mass flow under sonic (*M* = 1.0) conditions as:



$$\dot{m}(x) = \dot{m}*$$

$$\Rightarrow \quad \rho(x)\, A(x)\, v(x) = \rho^* A^* v^*$$

$$\Rightarrow \quad \frac{A(x)}{A^*} = \frac{\rho^*}{\rho(x)} \frac{v^*}{v(x)} \qquad \ldots. \qquad \ldots. \qquad \ldots.. \qquad (1)$$

Now,

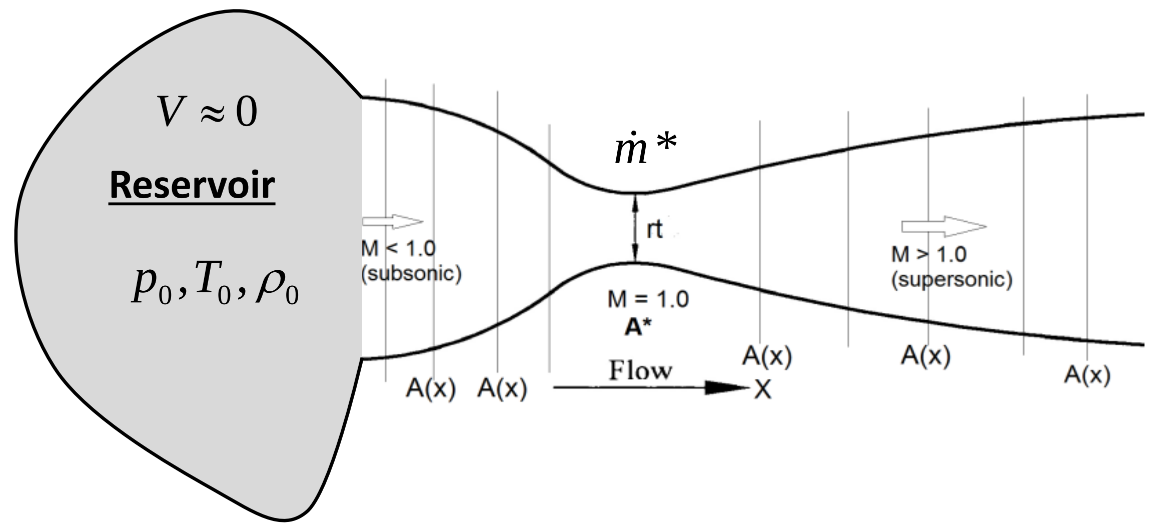
$$\frac{\rho^*}{\rho(x)} = \frac{\rho^*}{\rho_0} \times \frac{\rho_0}{\rho(x)}$$

$$= \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \times \left[1 + \frac{k-1}{2} M(x)^2\right]^{\frac{1}{k-1}}$$

$$\Rightarrow \frac{\rho^*}{\rho(x)} = \left\{\frac{2}{k+1}\left[1 + \frac{k-1}{2} M(x)^2\right]\right\}^{\frac{1}{k-1}}$$

at $M = 1.0$; $\quad \dfrac{\rho_0}{\rho^*} = \left(1 + \dfrac{k-1}{2} M^{*2}\right)^{\frac{1}{k-1}} \rightarrow \dfrac{\rho^*}{\rho_0} = \left(\dfrac{2}{k+1}\right)^{\frac{1}{k-1}}$

Perfect-Gas Area Changes



And

$$\frac{v^*}{v(x)} = \frac{a^*}{v(x)} \qquad \left[\text{at sonic point } M = 1.0 = \frac{v^*}{a^*}\right]$$

$$= \frac{\sqrt{kRT^*}}{v(x)}$$

$$= \frac{\sqrt{kRT(x)}}{v(x)}\left(\frac{T^*}{T_0}\right)^{\frac{1}{2}}\left(\frac{T_0}{T(x)}\right)^{\frac{1}{2}}$$

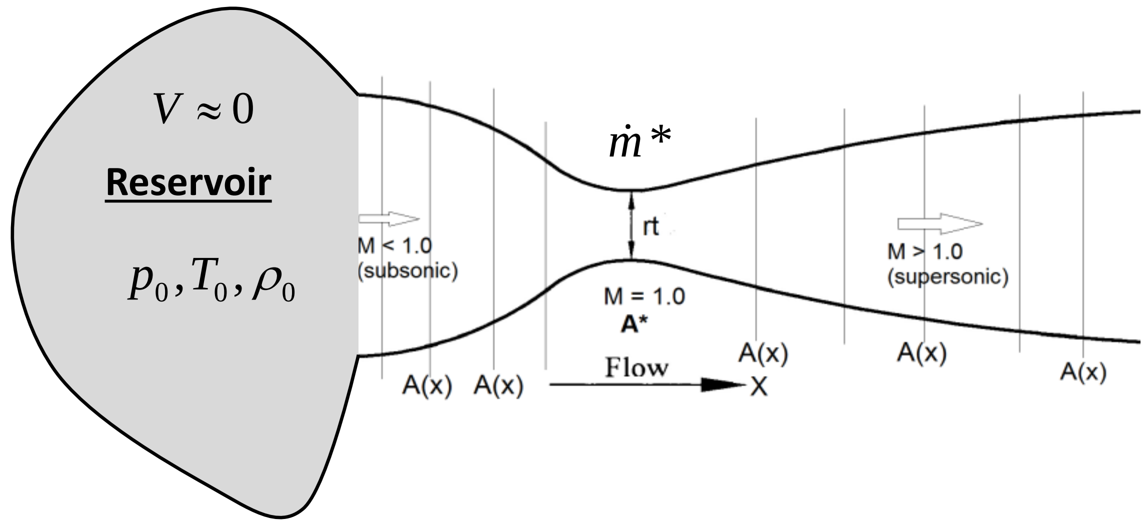
$$= \frac{1}{M(x)}\left(\frac{2}{k+1}\right)^{\frac{1}{2}}\left[1+\frac{k-1}{2}M(x)^2\right]^{\frac{1}{2}} \qquad ;\ M(x) = \frac{a(x)}{v(x)}$$

at $M = 1.0;\ \dfrac{T_0}{T^*} = \left(1+\dfrac{k-1}{2}M^{*2}\right) \rightarrow \dfrac{T^*}{T_0} = \left(\dfrac{2}{k+1}\right)$

$$\Rightarrow \frac{v^*}{v(x)} = \frac{1}{M(x)}\left\{\left(\frac{2}{k+1}\right)\left[1+\frac{k-1}{2}M(x)^2\right]\right\}^{\frac{1}{2}}$$

Perfect-Gas Area Changes

Use these two expressions in equation (1);



$$\frac{A(x)}{A^*} = \frac{1}{M(x)}\left[\frac{1+\frac{k-1}{2}M(x)^2}{\frac{k+1}{2}}\right]^{\frac{k+1}{2(k-1)}}$$

$$\Rightarrow \frac{A(x)}{A^*} = \frac{1}{M(x)}\left[\frac{2+(k-1)M(x)^2}{k+1}\right]^{\frac{k+1}{2(k-1)}} \quad \ldots\ldots \ldots\ldots (2)$$

Quantitative relation between area and velocity

This equation is to solve any one-dimensional isentropic gas flow problem given that the shape of the duct *A*(*x*) and the stagnation conditions are known and assuming that there are no shock waves in the duct (perfect/correct/ideal expansion).

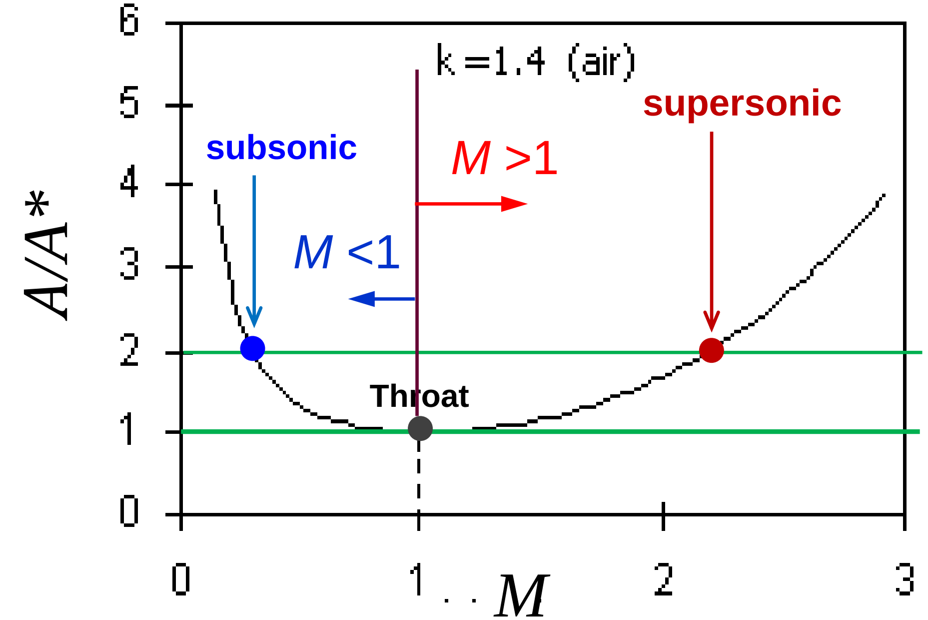
Perfect-Gas Area Changes

For air (k = 1.4), the equation (2) comes as

$$\frac{A(x)}{A^*} = \frac{1}{M(x)}\left[\frac{2+(k-1)M(x)^2}{k+1}\right]^{\frac{k+1}{2(k-1)}} \quad \text{.... ....(2)}$$

$$\boxed{\frac{A(x)}{A^*} = \frac{1}{M(x)}\frac{\left(1+0.2M(x)^2\right)^3}{1.728}}$$

expansion ratio, ε

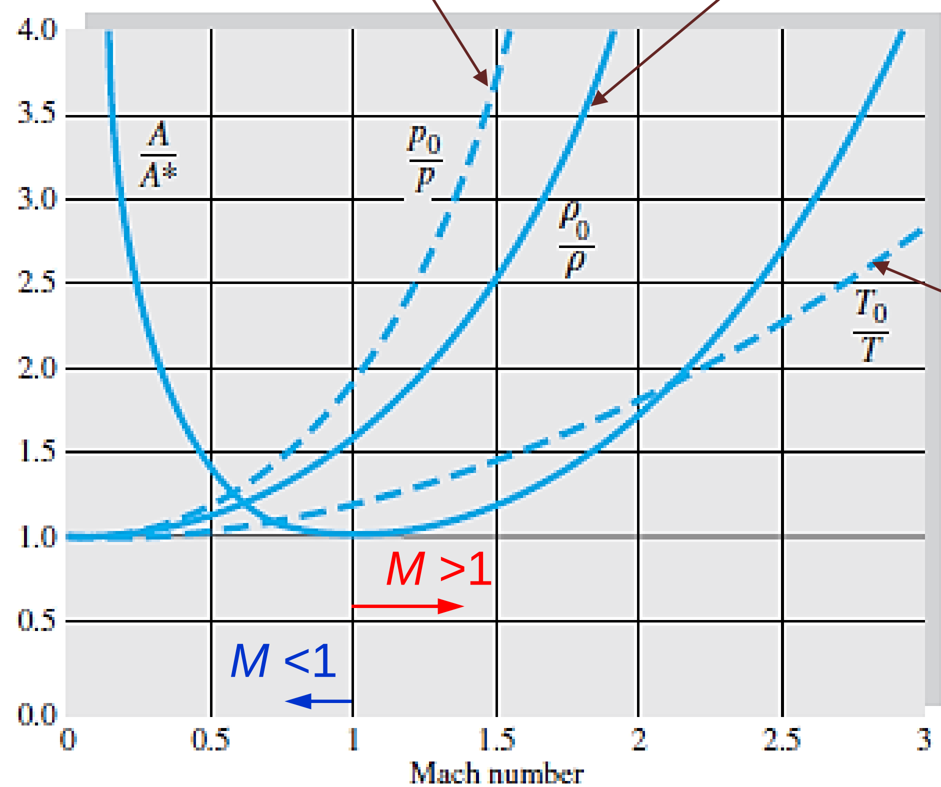


Area-Mach relation (isentropic, ideal)

- Figure shows that the minimum area that can occur in a given isentropic duct flow is sonic, or critical, throat area.
- **Each area ratio** ($A(x)/A^*$ i.e. expansion ratio) corresponds to the values of **two Mach number**.
- **One value is for subsonic flow case (M < 1) and other is for supersonic flow (M > 1).**

$p_0/p = \left(1 + \frac{k-1}{2} M^2\right)^{k/(k-1)}$

$\rho_0/\rho = \left(1 + \frac{k-1}{2} M^2\right)^{1/(k-1)}$



$T_0/T = \left(1 + \frac{k-1}{2} M^2\right)$

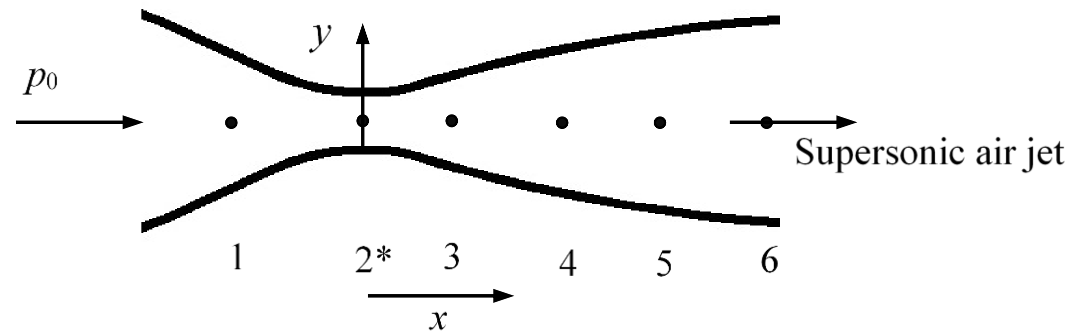
Fig. 9.7 Area ratio and fluid properties versus Mach number for isentropic flow of a perfect gas with $k = 1.4$.

Problem

A planar (*x*, *y*) convergent-divergent (C-D) nozzle is being used to expand the air to supersonic speed from a large reservoir as shown in Fig. The reservoir pressure and temperature are kept at 500 kPa and 300K, respectively Determine the Mach number, static pressure and temperature at the stations shown in the figure. Graphically present your results. Consider 1D isentropic flow in your calculation.

The location and dimension of the stations are given in the following table:

Station #	1	2*	3	4	5	6
height, *y* (mm)	25	20	22	25	28	31



Solution:

$$\frac{A(x)}{A^*} = \frac{1}{M(x)} \frac{\left(1+0.2M(x)^2\right)^3}{1.728} \qquad \rightarrow M(x)$$

$$\frac{p_0}{p(x)} = \left(1+\frac{k-1}{2}M(x)^2\right)^{\frac{k}{k-1}} \qquad \rightarrow \; p(x)$$

$$\frac{T_0}{T(x)} = \left(1+\frac{k-1}{2}M(x)^2\right) \qquad \rightarrow \; T(x)$$

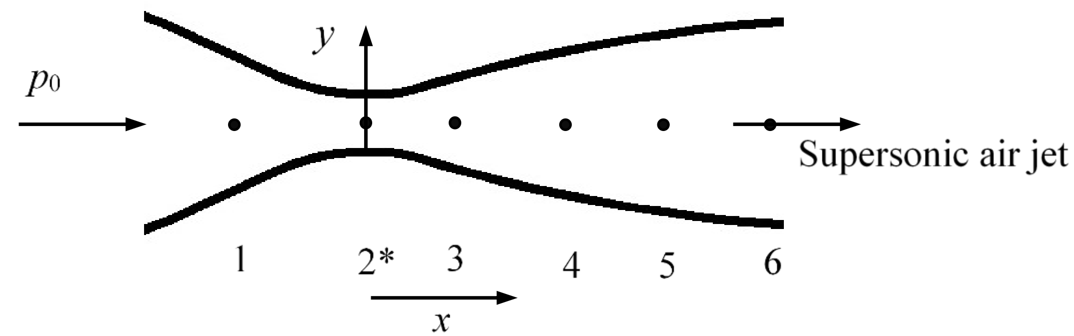
Plot $M(x)$ vs. x, $p(x)$ vs. x and $T(x)$ vs. x

Problem

An axisymmetric convergent-divergent (C-D) nozzle is being used to expand the air to supersonic speed from a large reservoir as shown in Fig. The reservoir pressure and temperature are kept at 500 kPa and 300K, respectively. Determine the Mach number, static pressure and temperature at the stations shown in the figure. Graphically present your results. Consider 1D isentropic flow in your calculation.

The location and dimension of the stations are given in the following table:

Station #	1	2*	3	4	5	6
diameter, *d* (mm)	25	20	22	25	28	31



Solution:

$$\frac{A(x)}{A^*} = \frac{1}{M(x)}\frac{\left(1+0.2M(x)^2\right)^3}{1.728} \quad \rightarrow M(x)$$

$$\frac{p_0}{p(x)} = \left(1+\frac{k-1}{2}M(x)^2\right)^{\frac{k}{k-1}} \quad \rightarrow \ p(x)$$

$$\frac{T_0}{T(x)} = \left(1+\frac{k-1}{2}M(x)^2\right) \quad \rightarrow \ T(x)$$

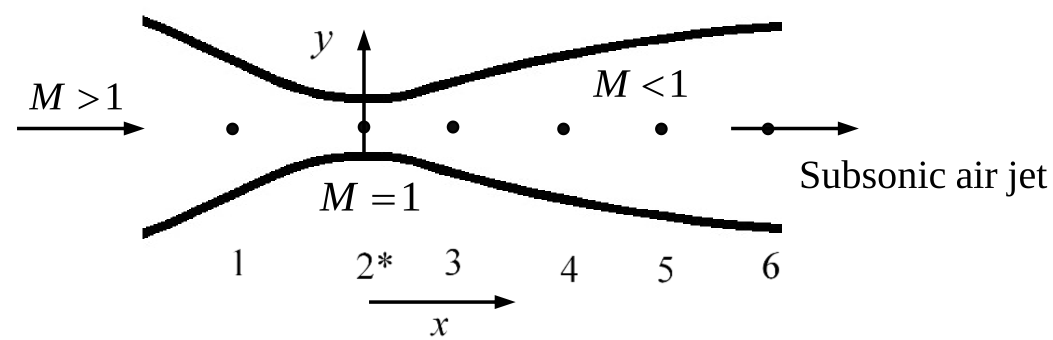
Plot $M(x)$ vs. x, $p(x)$ vs. x and $T(x)$ vs. x

Problem

An axisymmetric convergent-divergent (C-D) duct **is being feed supersonically ($M > 1$)** as shown in Fig. The static pressure and temperature at station # 1 are 30 kPa and 150K, respectively. Determine the Mach number, static pressure and temperature at the stations shown in the figure. Graphically present your results. Consider 1D isentropic flow in your calculation.

The location and dimension of the stations are given in the following table:

Station #	1	2*	3	4	5	6
diameter, *d* (mm)	25	20	22	25	28	31



Solution: **Duct will act as a supersonic diffuser.**

$$\frac{A(x)}{A^*} = \frac{1}{M(x)} \frac{\left(1+0.2M(x)^2\right)^3}{1.728} \quad \rightarrow M(x)$$

$$\boxed{\frac{p_0}{p_1} = \left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}} \quad \rightarrow \quad p_0}$$

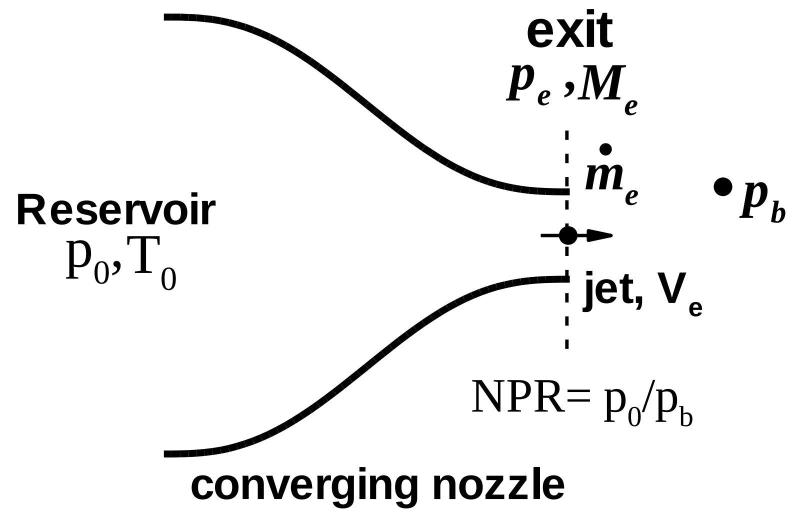
$$\frac{p_0}{p(x)} = \left(1+\frac{k-1}{2}M(x)^2\right)^{\frac{k}{k-1}} \quad \rightarrow \quad p(x)$$

$$\frac{T_0}{T(x)} = \left(1+\frac{k-1}{2}M(x)^2\right) \quad \rightarrow \quad T(x)$$

Plot $M(x)$ vs. x, $p(x)$ vs. x and $T(x)$ vs. x

Axial Thrust

Nozzles are used to develop axial **thrust** from high velocity jet from its exit.



Axial isentropic (ideal) thrust of the converging nozzle could be predicted by

$$T = \dot{m} V_e$$

$$\Rightarrow T = \rho_e A_e V_e \times V_e$$

$$\Rightarrow T = \rho_e A_e V_e^{\,2}$$

Exit jet velocity is to be determined: $V_e = ?$

Exit Jet Velocity

Consider two points in a flowing fluid: where 0: stagnation point ($V_0 = 0$) and other local point at exit ($V_e \neq 0$). The energy equation for 1-D isentropic flow-

$$\left(\frac{k}{k-1}\right)\frac{p}{\rho} + \frac{V^2}{2} = \text{constant}$$

$$\Rightarrow \left(\frac{k}{k-1}\right)\frac{p_e}{\rho_e} + \frac{V_e^2}{2} = \left(\frac{k}{k-1}\right)\frac{p_0}{\rho_0} + \frac{V_0^2}{2}$$

$$\Rightarrow \left(\frac{k}{k-1}\right)RT_e + \frac{V_e^2}{2} = \left(\frac{k}{k-1}\right)RT_0 + \frac{0^2}{2} \quad ; \quad p = \rho RT$$

$$\Rightarrow V_e^2 + \frac{2}{k-1}kRT_e = \frac{2}{k-1}kRT_0$$

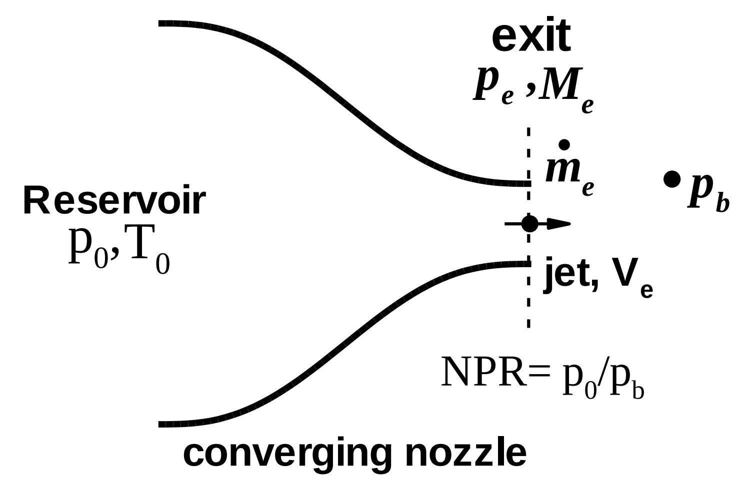
$$a_0 = \sqrt{kRT_0}$$

$$a_e = \sqrt{kRT_e}$$

$$\Rightarrow V_e^2 + \frac{2}{k-1}a_e^2 = \frac{2}{k-1}a_0^2$$

$$\Rightarrow V_e^2 = \frac{2}{k-1}a_0^2\left[1 - \left(\frac{a_e}{a_0}\right)^2\right]$$

$$\Rightarrow V_e = \left\{\frac{2}{k-1}a_0^2\left[1 - \left(\frac{a_e}{a_0}\right)^2\right]\right\}^{\frac{1}{2}}$$



Exit Jet Velocity

$$\Rightarrow V_e = \left\{ \frac{2}{k-1} a_0^2 \left[1 - \left(\frac{a_e}{a_0} \right)^2 \right] \right\}^{\frac{1}{2}}$$

$$\Rightarrow V_e = \left\{ \frac{2}{k-1} kRT_0 \left[1 - \frac{T_e}{T_0} \right] \right\}^{\frac{1}{2}}$$

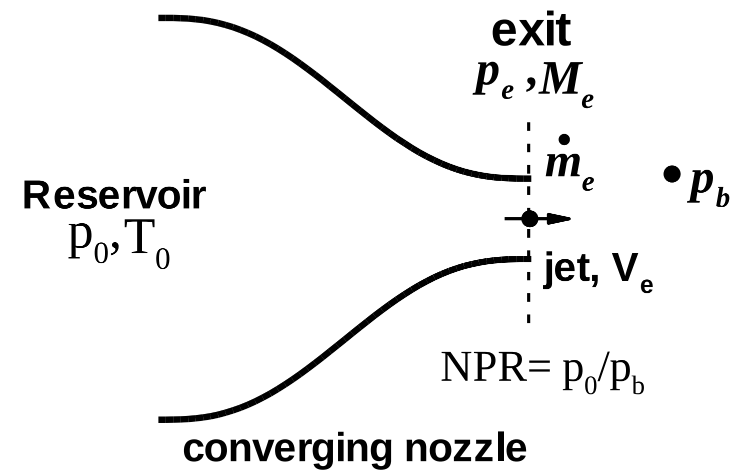
$$a_0 = \sqrt{kRT_0}$$

$$a_e = \sqrt{kRT_e}$$

$$\Rightarrow V_e = \left\{ \frac{2k}{k-1} RT_0 \left[1 - \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} \right] \right\}^{\frac{1}{2}}$$

From isentropic relation

$$; \quad \frac{T_e}{T_0} = \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} = \left(\frac{\rho_e}{\rho_0} \right)^{k-1}$$



Exit jet velocity is determined as:

$$\Rightarrow V_e = \left\{ \frac{2k}{k-1} RT_0 \left[1 - \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} \right] \right\}^{\frac{1}{2}}$$

Axial Thrust

Now the axial thrust comes as:

$$T = \dot{m}V_e$$

$$\Rightarrow T = \rho_e A_e V_e \times V_e$$

$$\Rightarrow T = \rho_e A_e V_e^{\,2}$$

$$\Rightarrow T = \rho_0 \frac{\rho_e}{\rho_0} A_e \left(\left\{ \frac{2k}{k-1} RT_0 \left[1 - \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} \right] \right\}^{\frac{1}{2}} \right)^2$$

$$V_e = \left\{ \frac{2k}{k-1} RT_0 \left[1 - \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} \right] \right\}^{\frac{1}{2}}$$

$$\Rightarrow T = \rho_0 \left(\frac{p_e}{p_0} \right)^{\frac{1}{k}} A_e \left\{ \frac{2k}{k-1} RT_0 \left[1 - \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} \right] \right\}$$

$$; \quad \frac{T_1}{T_0} = \left(\frac{p_1}{p_0} \right)^{\frac{k-1}{k}} = \left(\frac{\rho_1}{\rho_0} \right)^{k-1}$$

$$\Rightarrow \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} = \left(\frac{\rho_e}{\rho_0} \right)^{k-1}$$

$$\Rightarrow \frac{\rho_e}{\rho_0} = \left(\frac{p_e}{p_0} \right)^{\frac{1}{k}}$$

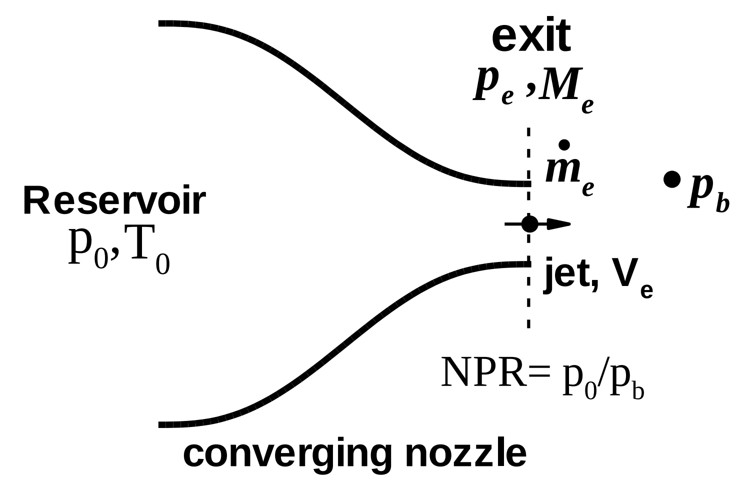
$$\Rightarrow T = \frac{p_0}{RT_0} \left(\frac{p_e}{p_0} \right)^{\frac{1}{k}} A_e \left\{ \frac{2k}{k-1} RT_0 \left[1 - \left(\frac{p_e}{p_0} \right)^{\frac{k-1}{k}} \right] \right\}$$

Ideal gas approx.

Axial Thrust

$$\Rightarrow T = p_0 \left(\frac{p_e}{p_0}\right)^{\frac{1}{k}} A_e \left\{ \frac{2k}{k-1} \left[1-\left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}\right] \right\}$$

Keeping the reservoir pressure, p_0 fixed; with the decrease of back pressure, p_b (or vice versa), the jet velocity will increase until the maximum jet velocity of $M = 1.0$. In all such cases, $p_e = p_b$



$$\Rightarrow T = p_0 \left(\frac{p_b}{p_0}\right)^{\frac{1}{k}} A_e \left\{ \frac{2k}{k-1} \left[1-\left(\frac{p_b}{p_0}\right)^{\frac{k-1}{k}}\right] \right\}$$

$$\Rightarrow T = p_0 \left(\frac{1}{\text{NPR}}\right)^{\frac{1}{k}} A_e \left\{ \frac{2k}{k-1} \left[1-\left(\frac{1}{\text{NPR}}\right)^{\frac{k-1}{k}}\right] \right\}$$

Nozzle pressure ratio, $\text{NPR} = \frac{p_0}{p_b}$

$$\Rightarrow \frac{T}{A_t} = p_0 \left(\frac{1}{\text{NPR}}\right)^{\frac{1}{k}} \left\{ \frac{2k}{k-1} \left[1-\left(\frac{1}{\text{NPR}}\right)^{\frac{k-1}{k}}\right] \right\}$$

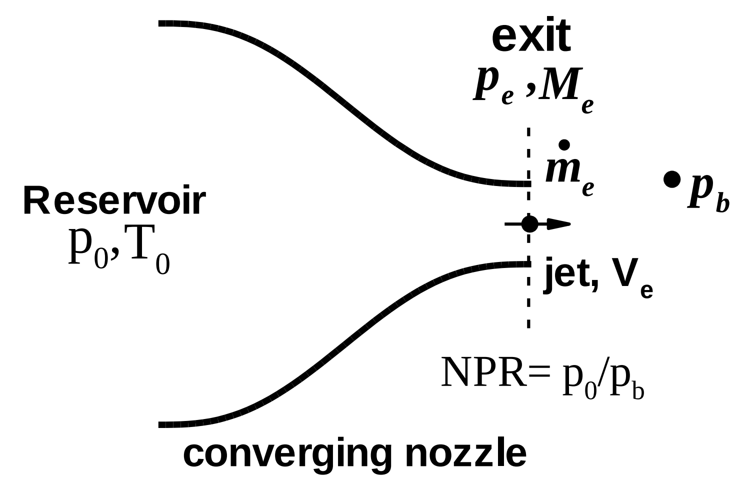
Isentropic Thrust per unit nozzle throat area

converging nozzle : $A_e = A_t$

Axial Thrust

For air ($k = 1.4$) and at critical condition (pressure ratio of 1.8929):

$$\Rightarrow \frac{T}{A_t} = p_0 \left(\frac{1}{1.8929}\right)^{\frac{1}{1.4}} \left\{ \frac{2(1.4)}{1.4-1} \left[1 - \left(\frac{1}{1.8929}\right)^{\frac{1.4-1}{1.4}} \right] \right\}$$



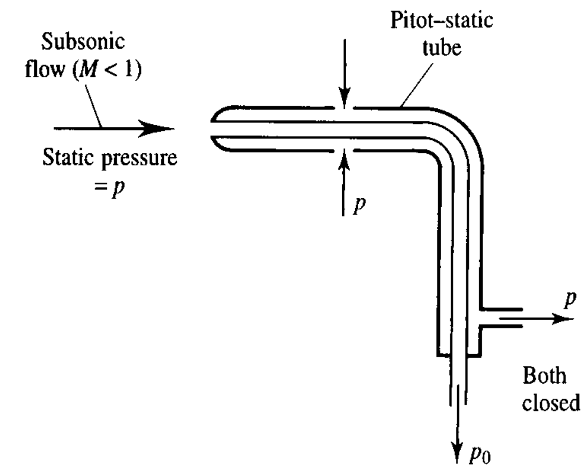
$$\Rightarrow \frac{T}{A_t} = 0.7395\ p_0$$

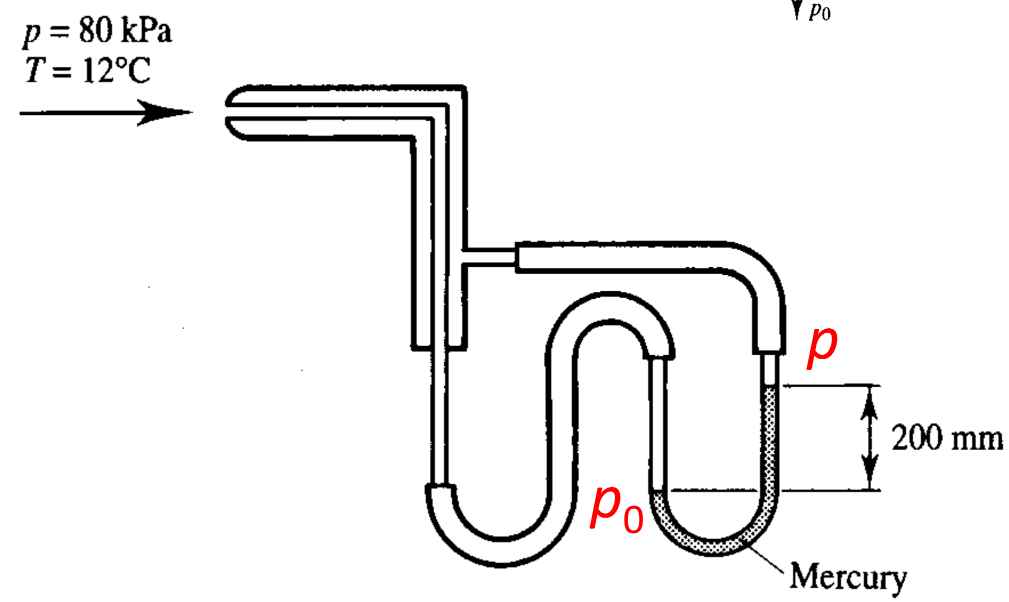
In case of converging nozzle, the maximum possible isentropic thrust per unit throat area is ≈ 74% of the reservoir pressure.

How about the remaining 26%?

Problem

A pitot-static tube is placed in a subsonic air flow. The static pressure and temperature in the flow are 80 kPa and 12°C respectively. The difference between the pitot and static pressures is measured using a manometer and found to be 200 mm of Hg. Find the air velocity and the Mach number.





Solution:

$$p_0 - p = \rho_m g \Delta Hm$$
$$= (13.6 \times 1000)(9.8)(200 \times 10^{-3}) = 26.68 \text{ kPa}$$

$$\therefore \frac{p_0 - p}{p} = \frac{26.68}{80}$$

$$\rightarrow \frac{p_0}{p} - 1 = 0.333 \qquad \therefore \frac{p_0}{p} = 1.333$$

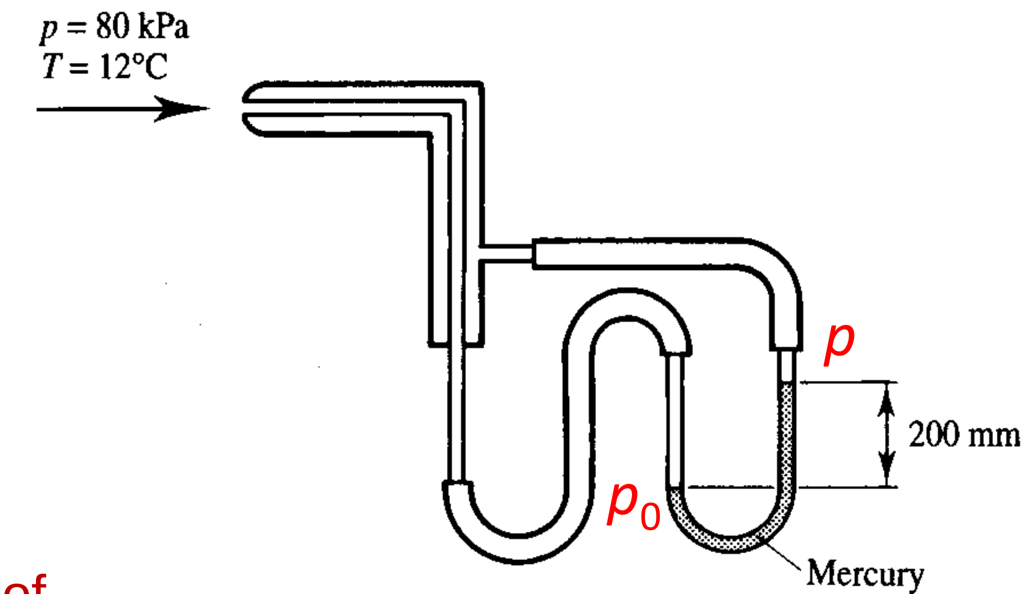
$$p_0/p = \left(1 + \frac{k-1}{2} M^2\right)^{k/(k-1)} \qquad \rightarrow \quad M = 0.654$$

$$\therefore V = Ma = (0.654)\sqrt{kR(273+12)} = 221.4\, m/s$$

Problem

However, using incompressible pitot-static tube relation:

$$V = \sqrt{\frac{2(p_0 - p)}{\rho}} = \sqrt{\frac{2(26.68 \times 10^3)}{\frac{80 \times 10^3}{(287)(273 + 12)}}} = 233.6\ m/s$$



Error 5.5 % !!

Use the correct equation to determine the flow velocity in case of compressible subsonic flow.

However, measurement in supersonic flow will be complicated due to formation of normal shock wave in front of the tube !!